BELYI MAPS AND DESSINS D'ENFANTS
LECTURE 1

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CONTENTS

I. ESSENTIAL RESULTS FROM COMPLEX ANALYSIS

I.1. Holomorphic functions.

Definition 1. A domain is an open, connected subset of $\mathbb{C}$.

Throughout this section, let $U \subseteq \mathbb{C}$ be an open set and $f : U \to \mathbb{C}$ be a function.

Definition 2. Let $U \subseteq \mathbb{C}$ be an open set. A function $f : U \to \mathbb{C}$ is differentiable at $z \in U$ if the limit

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists.

Remark 3. Here h is tending to 0 in $\mathbb{C}$! So this is a "two-dimensional" limit, just like in multivariable calculus.

Given a function $f : U \to \mathbb{C}$, we can write it as $f(z) = u(x,y) + iv(x,y)$, where $u, v : \mathbb{R}^2 \to \mathbb{R}$. The Cauchy-Riemann equations give a convenient way to check that a function is differentiable at a point.

Theorem 4. *Suppose that the partial derivatives u_x, u_y, v_x, v_y exist on U. If each of these partials is continuous at $z_0 \in U$ and if the Cauchy-Riemann equations*

$$u_x(z_0) = v_y(z_0) \qquad u_y(z_0) = -v_x(z_0) \tag{1}$$

are satisfied at z_0, then f is differentiable at z_0.

Definition 5. Suppose $U \subseteq \mathbb{C}$ is a nonempty open set and $f : U \to \mathbb{C}$. If f is differentiable at every point of U, then f is holomorphic on U. A function holomorphic on all of $\mathbb{C}$ is called entire.

Example 6. Polynomials, e^z, $\sin(z)$, and $\cos(z)$ are all entire. Rational functions are holomorphic wherever they are defined.

Date: April 20, 2021.

I.2. **Cauchy integral formula and consequences.**

Theorem 7 (Cauchy's Integral Theorem)**.** *Let $U \subseteq \mathbb{C}$ be a simply-connected open set, let $f : U \to \mathbb{C}$ be a holomorphic function, and let $\gamma : [0,1] \to U$ be a smooth closed curve. Then*

$$\int_\gamma f(z)\, dz = 0\,.$$

Theorem 8 (Cauchy Integral Formula)**.** *Let $\gamma : [0,1] \to \mathbb{C}$ be a simple closed curve oriented counterclockwise, and suppose that f is holomorphic on some domain U containing γ and its interior. Then*

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz$$

for any z_0 inside γ.

Remark 9. Here we are implicitly assuming the Jordan curve theorem.

Theorem 10 (Holomorphic implies infinitely differentiable)**.** *If $f : U \to \mathbb{C}$ is holomorphic, then so is f'. Moreover, f' is given by*

$$f'(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta$$

where γ is any simple closed curve in U oriented counterclockwise containing z in its interior.

Corollary 11. *If $f : U \to \mathbb{C}$ is holomorphic, then it is infinitely differentiable. Moreover,*

$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{k+1}}\, d\zeta$$

where γ is as above.

Theorem 12 (Holomorphic implies analytic)**.** *Suppose that $f : U \to \mathbb{C}$ is holomorphic and $z_0 \in U$ is a point such that the open disc $D := D(z_0, r) \subseteq U$. Then f is equal to its Taylor series on D, i.e.,*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for every $z \in D$, where $a_n = \dfrac{f^{(n)}(z_0)}{n!}$.

Theorem 13 (Liouville)**.** *The only bounded entire functions are constant functions.*

Theorem 14 (Identity Theorem)**.** *Let D be a domain, and suppose that $f, g : D \to \mathbb{C}$ are analytic. If $f(z) = g(z)$ for all z in some set S with a limit point in D, then $f = g$.*

Definition 15. Let $U \subseteq \mathbb{C}$ be an open set and $f : U \to \mathbb{C}$ be a function. If $f(V)$ is open for all open subsets $V \subseteq U$, then f is an open mapping.

Theorem 16 (Open Mapping Theorem)**.** *If a function f is analytic and nonconstant on a domain $D \subseteq \mathbb{C}$, then f is an open mapping on D. In particular, $f(D)$ is also a domain.*

II. Riemann surfaces

II.1. **Definitions and first examples.**

Definition 17. A topological surface X is a Hausdorff, second-countable space equipped with a collection of coordinate charts $\{(U_i, \varphi_i)\}_i$ where U_i is an open subset of X, and $\varphi_i : U_i \to \widehat{U_i}$ is a homeomorphism from U_i to an open subset $\widehat{U_i} = \varphi_i(U_i) \subseteq \mathbb{C}$ for each i, satisfying the following conditions.

1. $\{U_i\}_i$ is open cover of X, i.e., $\bigcup_i U_i = C$; and
2. Whenever $U_i \cap U_j \neq \varnothing$, the transition function
$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$
is a homeomorphism.

Such a collection of charts is called an atlas, and the inverse φ_i^{-1} of a coordinate map is called a parametrization.

Remark 18. I think condition (2) is actually redundant, but the terminology is useful to have.

Definition 19. A Riemann surface is a connected topological surface such that the transition functions of the atlas are holomorphic. Such an atlas is called a holomorphic atlas.

Remark 20. Other types of manifolds can be defined analogously by changing the requirement on the transition functions. E.g., a smooth surface is one whose transition functions are C^∞.

Example 21.

1. $\mathbb{C}$, or any open subset $U \subseteq \mathbb{C}$ is a Riemann surface. Moreover, they can be covered by a single chart. Two important example are the upper half-plane
$$\mathfrak{H} := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$$
and the open unit disc
$$\mathfrak{D} := \{z \in \mathbb{C} : |z| < 1\}.$$
2. The sphere $\mathbb{S}^2$. Let
$$\mathbb{S}^2 = \{(x, y, t) \in \mathbb{R}^3 : x^2 + y^2 + t^2 = 1\}.$$
We define a holomorphic atlas on $\mathbb{S}^2$ as follows. Let
$$U_1 = \mathbb{S}^2 \setminus \{(0,0,1)\}, \qquad \varphi_1(x,y,t) = \frac{x}{1-t} + i\frac{y}{1-t}$$
$$U_2 = \mathbb{S}^2 \setminus \{(0,0,-1)\}, \qquad \varphi_2(x,y,t) = \frac{x}{1+t} - i\frac{y}{1+t}.$$
(Note that these coordinate maps are the stereographic projections from the north and south pole, respectively.) One can compute that $(\varphi_2 \circ \varphi_1^{-1})(z) = 1/z$ where z is the varlable on $\varphi_1(U_1)$.

(3) The Riemann sphere $\widehat{\mathbb{C}}$. We define $\widehat{\mathbb{C}}$ topologically to be the one-point compactification of $\mathbb{C}$. This means that $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ as a set. We define the topology on $\widehat{\mathbb{C}}$ to consist of:

- the open subsets of $\mathbb{C}$; and
- the sets $(\mathbb{C} \setminus K) \cup \{\infty\}$, where $K \subseteq \mathbb{C}$ is a compact subset.

(So the family of sets $D(\infty, R) := \{z \in \mathbb{C} : |z| > R\} \cup \{\infty\}$ forms a neighborhood basis of ∞.)

We define an atlas on $\widehat{\mathbb{C}}$ by:

$$U_1 = \mathbb{C}, \qquad \varphi_1(z) = z$$
$$U_2 = \widehat{\mathbb{C}} \setminus \{0\}, \qquad \varphi_2(z) = \begin{cases} 1/z & \text{if } z \neq \infty \\ 0 & \text{if } z = \infty \,. \end{cases}$$

(4) The projective line $\mathbb{P}^1$. Define

$$\mathbb{P}^1 = \frac{\mathbb{C}^2 \setminus \{(0,0)\}}{\sim}$$

where $\sim$ is the equivalence relation defined by: given $v \in \mathbb{C}^2 \setminus \{(0,0)\}$, $v \sim \lambda v$ for all $\lambda \in \mathbb{C}^\times$. Given $(z_0, z_1) \in \mathbb{C}^2$, denote its equivalence class in $\mathbb{P}^1$ by $[z_0 : z_1]$. Thus $[z_0 : z_1] = [\lambda z_0 : \lambda z_1]$ for all $\lambda \in \mathbb{C}^\times$.

We define an atlas on $\mathbb{P}^1$ by:

$$U_0 = \{[z_0 : z_1] \in \mathbb{P}^1 : z_0 \neq 0\}, \qquad \varphi_0([z_0 : z_1]) = \frac{z_1}{z_0}$$
$$U_1 = \{[z_0 : z_1] \in \mathbb{P}^1 : z_1 \neq 0\}, \qquad \varphi_1([z_0 : z_1]) = \frac{z_0}{z_1}$$

(5) A complex torus. Fix $\omega_1, \omega_2 \in \mathbb{C}$ that are $\mathbb{R}$-linearly independent, and let $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ be the lattice they span. (A good example to keep in mind is $\Lambda = \mathbb{Z}[i] = \mathbb{Z} \oplus i\mathbb{Z}$.) Then Λ acts on $\mathbb{C}$ by addition. Let X be the quotient $\mathbb{C}/\Lambda$, whose elements are orbits:

$$[z] = z + \Lambda = \{z + \omega : \omega \in \Lambda\} \,.$$

We equip $\mathbb{C}/\Lambda$ with the quotient topology, so $U \subseteq \mathbb{C}/\Lambda$ is open iff $\pi^{-1}(U) \subseteq \mathbb{C}$ is open, where $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ is the quotient map $z \mapsto z + \Lambda$.

Given an open subset $U \subseteq X$, then $\pi(\pi^{-1}(U))$ so U is the image of an open subset of U under π. Conversely, π is an open map: given $V \subseteq \mathbb{C}$ open, then

$$\pi^{-1}(\pi(V)) = \bigcup_{\omega \in \Lambda} (\omega + V) \,.$$

Each $\omega + V$ is the translate of an open set, so $\pi^{-1}(\pi(V))$ is a union of open sets, hence is open.

We now define an atlas on X. First, note that since Λ is discrete, there there exists $\epsilon > 0$ such that $|\omega| > 2\epsilon$ for all $0 \neq \omega \in \Lambda$. (For instance, in the case $\Lambda = \mathbb{Z}[i]$ we could take any $\epsilon < 1/2$.) Fix such an ϵ. Given $s \in \mathbb{C}$, let $D_s = D(s, \epsilon)$. Note that all elements of D_s are distinct mod Λ.

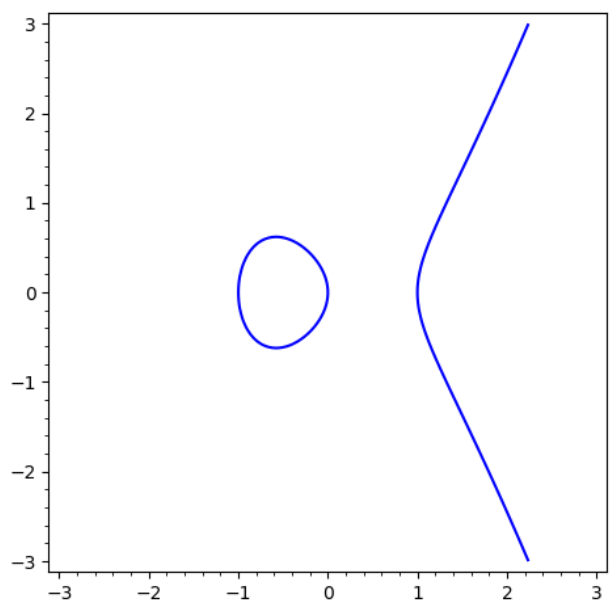
We claim that for any $s \in \mathbb{C}$ and any such ϵ, $\pi|_{D_s} : D_s \to \pi(D_s) \subseteq \mathbb{C}/\Lambda$ is a homeomorphism. It is continuous, surjective, and open because π is, and it is also injective by the choice of ϵ. For each $s \in \mathbb{C}$, define $\varphi_s : \pi(D_s) \to D_s$ to be $(\pi|_{D_s})^{-1}$. Then $\{(D_s, \varphi_s) : s \in \mathbb{C}\}$ is an atlas on X, so X is a topological surface. One can even show that this gives a holomorphic atlas (which I'll probably give as homework), so X is a Riemann surface.

Remark 22. The textbook glosses over an important point. Say we take our atlas and throw in another open subset that is contained in one of the charts in our atlas, equipping them with the restriction of the coordinate map. Is this a different Riemann surface? According to our definition, yes, but intuitively we would think these two atlases are equivalent in some sense.

The notion of the equivalence of two atlases is treated carefully in Miranda's *Algebraic Curves and Riemann Surfaces*. The upshot is that every atlas is contained in a unique maximal atlas.

All of these examples seem very topological, but many important examples arise from algebraic geometry. We'll pursue this idea more systematically next time, but let's just try to build some intuition by looking at a concrete example.

Example 23. Let $X = \{(x, y) \in \mathbb{C}^2 : y^2 = x^3 - x\}$. Here's a graph of its real points.



Let $f(x) = x^3 - x = x(x-1)(x+1)$ and let $F(x, y) = y^2 - f(x)$. Given a point $P = (x_0, y_0)$ such that the tangent line at P isn't vertical, i.e., whenever $F_y(P) = 2y_0 \neq 0$, we can restrict the projection $(x, y) \mapsto x$ onto the x-axis to a neighborhood of P in order to define a coordinate chart at P. When $F_y(P) = y_0 = 0$, we can instead restrict the projection $(x, y) \mapsto y$ onto the y-axis to define a coordinate chart at P.

Remark 24. The idea above can be generalized and made rigorous using the Inverse Function Theorem, which we'll discuss next time.